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Integral Operators and the First Initial Boundary Value Problem for Pseudoparabolic Equations with Analytic Coefficients*

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I. INTRODUCTION

This paper was motivated by the desire to derive constructive methods for solving the first initial boundary value problem for the equation

$$p_t - \eta \Delta_n p_t = \kappa \Delta_n p; \qquad n = 2, 3, \tag{1.1}$$

where $p_t = \partial p/\partial t$ and κ and η are positive constants. (Equation (1.1) has previously been studied for the case $n = 1$ in [4] and will not be treated here.) This equation arises, for example, in the theory of seepage of liquids in fissured rocks [1], in whlch case p denotes the pressure in the fissures and η and κ are constants determined by the physical properties of the rock. The specific problem which arises is to construct a solution of Eq. (1.1) which assumes given initial conditions at $t = 0$ and prescribed boundary values on the cylinder $\partial D \times T$ (where D is a simply connected domain in $\mathbb{R}^n$ with Lyapunov boundary ∂D and $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a fixed, but arbitrarily large, positive constant). As in the theory of seepage in a porous medium, the steady-state initial conditions are of greatest interest (i.e., the harmonic initial distributions $p^{(1)}$ which satisfy Eq. (1.1)). Setting

$$p = e^{-(\kappa/\eta)t}u + p^{(1)}, \tag{1.2}$$

it is, therefore, seen that without loss of generality we can consider the equation,

$$\Delta_n u_t - (1/\eta)\, u_t + (\kappa/\eta^2)\, u = 0, \tag{1.3}$$

and assume that $u = 0$ at $t = 0$.

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In this paper we will consider the more general problem of solving the first initial boundary value problem (with homogeneous initial conditions) for the equations

$$\Delta_2 u_t + c(x, y)\, u_t + d(x, y) u = 0, \tag{1.4}$$

$$\Delta_n u_t + A(r^2)\, u_t + B(r^2) u = 0; \qquad n \geqslant 2, \tag{1.5}$$

where $c(x, y)$ and $d(x, y)$ are real valued (for x and y real) entire functions of their independent (complex) variables and $A(r^2)$ and $B(r^2)$ are real valued entire functions of $r^2 = x_1^2 + \cdots + x_n^2$. Our basic goal is to reduce the problem of finding a solution of the first initial boundary value problem for Eqs. (1.4) and (1.5) to that of solving an integral equation. For Eq. (1.4) this is accomplished by using the fundamental solution which was previously constructed by the author in [5]. However, for Eq. (1.5) a fundamental solution has not yet been constructed and we adopt an approach based on the use of integral operators. This involves first constructing integral operators for Eq. (1.4) which are analogous to Bergman's operators for elliptic equations [2, 9] and then using this as a basis for a "method of ascent" [6, 10, 11] to construct integral operators for Eq. (1.5). The result is an integral operator which maps solutions of the equation,

$$\Delta_n u_t = 0, \tag{1.6}$$

onto solutions of Eq. (1.5), and through the use of such an operator it is possible to reduce the first initial boundary value problem for Eq. (1.5) to the problem of solving an integral equation. An interesting aspect of our analysis is that the integral equations which arise are of neither Fredholm nor Volterra type, but of the form $f = (\mathbf{I} + \mathbf{T} + \mathbf{L})\mu$ where $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator. We will show that under the assumption that $c(x, y) \leqslant 0$ and $A(r^2) \leqslant 0$, respectively, such equations are always solvable.

Equations of the form of Eqs. (1.4) and (1.5) were first systematically studied by Sobolev [18] and Galpern [7], and more recently by Showalter [13–16], Ting [20] and Showalter and Ting [17], who refer to such equations as being of pseudoparabolic or Sobolev–Galpern type. The approach of these authors is based on Hilbert space methods and yield quite general existence and uniqueness theorems in $\mathbb{R}^n \times T$ for n an arbitrary integer. In the case of one space dimension ($n = 1$) these equations have also been studied through the use of integral operators [4], Laplace transforms [19], and separation of variables [3]. In addition, the analytic theory of pseudoparabolic equations in two space dimensions has been studied by Colton [5].

II. Bergman Operators

We will now construct an integral operator which maps analytic functions of a single complex variable depending on a parameter t onto the class of real valued strong solutions of Eq. (1.4) which vanish at $t = 0$. Let D be a simply connected domain of $\mathbb{R}^2$ and $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Then from [5] we have that strong solutions of Eq. (1.4) defined in $D \times T$ and vanishing at $t = 0$ are, for each fixed t, analytic functions of $z = x + iy$ and $z^* = x - iy$ in $D \times D^*$ where $D^* = \{z^*: \bar{z}^* \in D\}$. Hence, we can rewrite Eq. (1.4) in the form

$$U_{zz^*t} + C(z, z^*)\, U_t + D(z, z^*)U = 0, \tag{2.1}$$

where

$$\begin{aligned} U(z, z^*, t) &= u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}, t\right), \\ C(z, z^*) &= \frac{1}{4} c\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right), \\ D(z, z^*) &= \frac{1}{4} d\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right). \end{aligned} \tag{2.2}$$

We now look for a solution of Eq. (2.1) in the form

$$U(z, z^*, t) = \int_0^t \int_{-1}^{+1} E(z, z^*, t - \tau, s) f_\tau\left(\frac{z}{2}(1 - s^2), \tau\right) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{2.3}$$

where $f(z, t)$ is an analytic function of z and continuously differentiable with respect to t. Without loss of generality we can assume that $f(z, 0) = 0$. Substituting (2.3) into (2.1) and integrating by parts (c.f. [2, 9]) shows that $E(z, z^*, t, s)$ must satisfy the singular partial differential equation

$$(1 - s^2)\, E_{z^*st} - (1/s)\, E_{z^*t} + 2sz(E_{zz^*t} + CE_t + DE) = 0, \tag{2.4}$$

provided we impose the boundary conditions

$$\begin{aligned} E(z, z^*, 0, s) &= 0, \\ E_{z^*t}(0, z^*, t, s) &= 0, \\ E_{z^*t}(z, z^*, t, 0) &= 0, \end{aligned} \tag{2.5}$$

and require $E(z, z^*, t, s)$ to be analytic for $t \in T$, $s \in I = \{s: |\, s\,| \leqslant 1\}$, and $(z, z^*) \in D \times D^*$. Following Bergman [2] we now look for a solution of Eq. (2.4) in the form

$$E(z, z^*, t, s) = t + \sum_{k=1}^{\infty} s^{2k} z^k \int_0^{z^*} P^{(2k)}(z, \zeta^*, t)\, d\zeta^*, \tag{2.6}$$

where we require $P^{(2k)}(z, z^*, 0) = 0$ for $k = 1, 2, \ldots$. Substituting (2.6) into (2.4) and integrating with respect to t yields the following recursion formulas for the $P^{(2k)}(z, z^*, t)$:

$$\begin{aligned} P^{(2)} &= -2tC - t^2D, \\ (2k+1)\, P^{(2k+2)} &= -2 \Big[P_z^{(2k)} + C \int_0^{z^*} P^{(2k)}(z, \zeta^*, t)\, d\zeta^* \\ &\quad + D \int_0^t \int_0^{z^*} P^{(2k)}(z, \zeta^*, \tau)\, d\zeta^*\, d\tau \Big]; \quad k \geqslant 0. \end{aligned} \tag{2.7}$$

Hence, each of the $P^{(2k)}$, $k = 1, 2, \ldots$, is uniquely determined. We now must show that the series (2.6) converges uniformly in $D \times D^* \times T \times I$. We will do a bit more than this and show that due to the fact that $C(z, z^*)$ and $D(z, z^*)$ are entire functions of z and z^* the series (2.6) converges for arbitrary values of z and z^* (uniformly on compact subsets in the space of two complex variables). Let r be an arbitrarily large positive number and let C_0 be a positive constant chosen such that (as functions of z)

$$\begin{aligned} C(z, z^*) &\ll \frac{C_0}{1 - (z/r)}, \\ D(z, z^*) &\ll \frac{C_0}{1 - (z/r)}, \end{aligned} \tag{2.8}$$

for $|z| < r$, $|z^*| < r$, where "$\ll$" denotes domination (c.f. [2]). We will now show by induction that there exist positive constants M and δ which are independent of k such that for $|z| < r$, $|z^*| < r$, $|t| \leqslant t_0$,

$$P^{(2k)} \ll M 2^k (1 + \delta)^k (2k - 1)^{-1} [1 - (z/r)]^{-(2k-1)} r^{-k}. \tag{2.9}$$

From Eqs. (2.7) and (2.8) this is obviously true for $k = 1$. Now suppose for $k = j$ we have

$$P^{(2j)} \ll M_j 2^j (1 + \delta)^j (2j - 1)^{-1} [1 - (z/r)]^{-(2j-1)} r^{-j}, \tag{2.10}$$

where for the time being we allow M_j to depend on j. Then from Eqs. (2.7) and (2.8) and the standard use of the theory of dominants we have

$$P^{(2j+2)} \ll \frac{M_j 2^{j+1} (1 + \delta)^j}{2j + 1} \left\{ 1 + \frac{C_0 r^2 + C_0 r^2 t_0}{2j - 1} \right\} [1 - (z/r)]^{-(2j+1)}\, r^{-j-1} \tag{2.11}$$

(the main property of dominants we have used in deriving Eq. (2.11) is that if $f \ll g$ then $f \ll g[1 - (z/r)]^{-1}$). By setting

$$M_{j+1} = M_j (1 + \delta)^{-1} \left\{ 1 + \frac{C_0 r^2 + C_0 r^2 t_0}{2j - 1} \right\}, \tag{2.12}$$

we have shown that Eq. (2.10) is true for j replaced by $j + 1$. But for j sufficiently large we have $M_{j+1} \leqslant M_j$, i.e., there exists a positive constant M which is independent of j such that $M_j \leqslant M$ for all j. Equation (2.9) is now established for all k.

We now return to the convergence of the series (2.6). First, let $D_{\alpha r} = \{z: |z| < r/\alpha\}$ and $D_{\alpha r}^* = \{z^*: |z^*| < r/\alpha\}$ where $\alpha > 1$ is fixed. We will show that for α sufficiently large the series (2.6) converges in $D_{\alpha r} \times D_{\alpha r}^* \times T \times I$. Using the estimate $[1 - (|z|/r)] \geqslant (\alpha - 1/\alpha)$ and the fact that if $f \ll M[1 - (z/r)]^{-1}$ then $|f| \leqslant M[1 - (|z|/r)]^{-1}$ we have from Eq. (2.9) that the series (2.6) is majorized in $D_{\alpha r} \times D_{\alpha r}^* \times T \times I$ by

$$t_0 + \sum_{k=1}^{\infty} \frac{rM2^k(1 + \delta)^k \alpha^{k-1}}{(2k-1)(\alpha-1)^{2k-1}}. \tag{2.13}$$

If α is chosen such that $2(1 + \delta)\,\alpha(\alpha - 1)^{-2} < 1$ then the series (2.13) converges. Since r is an arbitrarily large positive number and δ is arbitrarily small and independent of r, we now have that the series (2.6) converges absolutely and uniformly for $|z| < r$, $|z^*| < r$, $|t| \leqslant t_0$ and $|s| \leqslant 1$.

We have now proved that the operator defined by Eq. (2.3) exists and maps analytic functions $f(z, t)$ into the class of (complex valued) strong solutions of Eq. (1.4) with homogeneous initial data. It is important for our purposes that this mapping, in fact, be onto the class of real valued solutions of Eq. (1.4). However, this is obviously not the case (even if we take the real part of the right side of Eq. (2.3)) since if $U(z, z^*t)$ can be represented in the form of Eq. (2.3), it must be true that $U_t(z, z^*, 0) = 0$. There are obviously solutions of Eq. (1.4) which do not satisfy this property (c.f. [5]). However, if we note that if $u(x, y, t)$ is a solution of Eq. (1.4) then so is $u_t(x, y, t)$, we can differentiate Eq. (2.3) with respect to t, take the real part of both sides, and arrive at a new operator which maps analytic functions onto solutions of Eq. (1.4) with homogeneous initial data. The step of taking the real part of both sides of Eq. (2.3) is justified since $c(x, y)$ and $d(x, y)$ are real valued for x and y real. We are, thus, led to consider the operator defined (for x, y real) by

$$u(x, y, t) = \operatorname{Re} \int_0^t \int_{-1}^{+1} E_t(z, \bar{z}, t - \tau, s) f_\tau\left(\frac{z}{2}(1 - s^2), \tau\right) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{2.14}$$

where "Re" denotes "take the real part." We will now show that every real valued strong solution $u(x, y, t)$ of Eq. (1.4) with homogeneous initial data can be represented in the form of Eq. (2.14). An elementary power series analysis (c.f. [21, pp. 55–56]) coupled with the results of [5] shows that such solutions are uniquely determined by their values on the characteristic $z^* = 0$.

Extending Eq. (2.14) to complex values of x and y and evaluating at $z^* = 0$ leads to the equation

$$U(z, 0, t) = \frac{1}{2} \int_{-1}^{+1} f\left(\frac{z}{2}(1 - s^2), t\right) \frac{ds}{(1 - s^2)^{1/2}} + \frac{\pi}{2} \bar{f}(0, t), \qquad (2.15)$$

where $\bar{f}(z, t) = \overline{f(\bar{z}, \bar{t})}$ and we have used the fact that $f(z, 0) = 0$. Equation (2.15) shows that $f(z, t)$ can be chosen such that $U(z, 0, t)$ assumes prescribed values (c.f. [2, pp. 12–13]), and, thus, $u(x, y, t) = U(z, \bar{z}, t)$ can be represented in the form of Eq. (2.14). Finally, we note that by comparing the recursion formulas (2.7) with those of Bergman [2, p. 13] we have

$$E_t(z, z^*, 0, s) = \tilde{E}(z, z^*, s), \qquad (2.16)$$

where $\tilde{E}(z, z^*, s)$ is Bergman's generating function for the elliptic equation $\Delta_2 u + c(x, y)u = 0$ (c.f. [2, 9]). We summarize the results of this section in the following theorem.

THEOREM 2.1. *Let $u(x, y, t)$ be a real valued strong solution of Eq. (1.4) defined in a cylindrical domain $D \times T$ where D is simply connected and suppose $u(x, y, 0) = 0$. Then $u(x, y, t)$ can be represented in the form of Eq. (2.14) where $f(z, t)$ is an analytic function of z and continuously differentiable with respect to t such that $f(z, 0) = 0$ and $E(z, z^*, t, s)$ is defined by Eqs. (2.6) and (2.7). $E(z, z^*, t, s)$ is an entire function of z and z^* and is analytic in t and s for $|t| \leqslant t_0$ and $|s| \leqslant 1$. $E_t(z, z^*, 0, s) = \tilde{E}(z, z^*, s)$ where $\tilde{E}(z, z^*, s)$ is Bergman's generating function for the elliptic equation $\Delta_2 u + c(x, y)u = 0$.*

The operator defined by Eq. (2.14) can be used to construct a complete family of solutions for Eq. (1.4). This is accomplished by setting $f(z, t) = z^l t^k$ and letting l and k be arbitrary nonnegative integers. Such a complete family can be used to approximate solutions of the first initial boundary value problem for Eq. (1.4) satisfying homogeneous initial conditions. We note that the operator presented in [5] can also be used to construct a complete family of solutions. However, the operator derived here is considerably easier to construct and is, therefore, more suitable for computational purposes.

III. THE METHOD OF ASCENT

We will now use the ideas of [6, 10] to extend the results of Section 2 to include Eq. (1.5). We first consider the case when $n = 2$. In this situation it is easily verified (c.f. [2, pp. 27–28]) that $E_t(z, \bar{z}, t, s)$ depends only on $r^2 = z\bar{z}$, t and s, and, hence, we can rewrite Eq. (2.14) in the form

$$u(x, y, t) = \int_0^t \int_{-1}^{+1} E_t(r^2, t - \tau, s)\, H_\tau[x(1 - s^2)^{1/2}, y(1 - s^2)^{1/2}, \tau] \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{3.1}$$

where

$$H_t(x, y, t) = \operatorname{Re} f_t[(z/2), t] \tag{3.2}$$

is a harmonic function of x and y for each fixed t, i.e. $H(x, y, t)$ is a solution of the pseudoparabolic equation (1.6) for $n = 2$. From the previously imposed condition that $f(z, 0) = 0$ we have that $H(x, y, 0) = 0$. From Eqs. (2.4)–(2.6) it can be shown that $E(r^2, t, s)$ satisfies the partial differential equation

$$(1 - s^2)\, E_{rst} - (1/s)\, E_{rt} + rs[E_{rrt} + (1/r)\, E_{rt} + AE_t + BE] = 0; \tag{3.3}$$

the initial conditions

$$\begin{aligned} E(r^2, 0, s) &= 0, \\ E_t(0, t, s) &= 1; \end{aligned} \tag{3.4}$$

and has a series expansion of the form

$$E(r^2, t, s) = t + \sum_{k=1}^{\infty} e^{(k)}(r^2, t)\, s^{2k}, \tag{3.5}$$

which converges absolutely and uniformly for $|s| \leqslant 1$ and r and t arbitrarily large (but bounded).

Now define $h(x, y, t)$ by

$$h(x, y, t) = \int_{-1}^{+1} H[x(1 - s^2)^{1/2}, y(1 - s^2)^{1/2}, t] \frac{ds}{(1 - s^2)^{1/2}}. \tag{3.6}$$

Then Eq. (3.1) can be rewritten (c.f. [6, 10]) as

$$u(x, y, t) = h(x, y, t) + \int_0^t \int_0^1 \sigma G_t(r^2, 1 - \sigma^2, t - \tau)\, h_\tau(x\sigma^2, y\sigma^2, \tau)\, d\sigma\, d\tau, \tag{3.7}$$

where $h(x, y, t)$ is again a solution of $\Delta_2 u_t = 0$ satisfying the initial condition $h(x, y, 0) = 0$ and $G(r^2, \rho, t)$ is defined by

$$G(r^2, \rho, t) = \sum_{k=1}^{\infty} \frac{2e^{(k)}(r^2, t)\, \Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\, \Gamma(k)}\, \rho^{k-1}. \tag{3.8}$$

From the analysis of section two it is clear that Eq. (3.7) defines a mapping of the class of real valued solutions of the equation $\Delta_2 u_t = 0$ which vanish at $t = 0$ and are defined in a domain $D \times T$ (where D is starlike with respect to the origin) onto the class of real valued solutions of Eq. (1.5) (for $n = 2$) which vanish at $t = 0$ and are defined in $D \times T$.

We now want to generalize the representation (3.7) from $n = 2$ to general n. To this end we first look for solutions of Eq. (1.5) in the form

$$u(\mathbf{x}, t) = \int_0^t \int_0^1 s^{n-2} E(r^2, t - \tau, s; n)\, H_\tau(\mathbf{x}(1 - s^2), \tau) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \qquad (3.9)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $H(\mathbf{x}, t)$ is a real valued solution of Eq. (1.6) which vanishes at $t = 0$. We require that $E(r^2, t, s; n)$ be an entire function of r^2 and t, analytic in s for $|s| \leqslant 1$, and satisfy the initial conditions $E(r^2, 0, s; n) = 0$, $E_t(0, t, s; n) = 1$. We now temporarily replace the path of integration from zero to one by a loop starting from $s = +1$, passing counterclockwise around the origin and onto the second sheet of the Riemann surface of the integrand, and then back up to $s = +1$, and substitute the resulting expression into the differential equation (1.5). If $u(\mathbf{x}, t)$ is to be a solution of Eq. (1.5) it is then easily verified by integrating by parts that $E(r^2, t, s; n)$ must satisfy the singular partial differential equation

$$(1 - s^2) E_{rst} + (n - 3/s) E_{rt} + rs[E_{rrt} + (1/r) E_{rt} + AE_t + BE] = 0. \qquad (3.10)$$

We now look for a solution of Eq. (3.10) in the form

$$E(r^2, t, s; n) = t + \sum_{k=1}^{\infty} e^{(k)}(r^2, t; n)\, s^{2k}. \qquad (3.11)$$

Substituting (3.11) into Eq. (3.9) and making use of the initial condition $E(r^2, 0, s; n) = 0$ yields the following recursion formulas for the determination of the $e^{(k)}(r^2, t; n)$:

$$\begin{aligned} (n - 1)\, e_r^{(1)} &= -trA - (t^2/2)\, rB \\ (2k + n - 3)\, e_r^{(k)} &= (2k - 3)\, e_r^{(k-1)} - re_{rr}^{(k-1)} - rAe^{(k-1)} \\ &\quad -rB \int_0^t e^{(k-1)}\, d\tau; \quad k \geqslant 2. \end{aligned} \qquad (3.12)$$

From the initial condition $E_t(0, t, s; n) = 1$ we have the initial conditions

$$e^{(k)}(0, t; n) = 0; \qquad k = 1, 2, \dots, \qquad (3.13)$$

for each of the $e^{(k)}(r^2, t; n)$. Hence, each of the $e^{(k)}(r^2, t; n)$ in Eq. (3.11) is uniquely determined. We must now show the series (3.11) converges uniformly for r and t arbitrarily large (but bounded) and $|s| \leqslant 1$. We first note that for $n = 2$ the $e^{(k)}(r^2, t; 2)$ are identical with the function $e^{(k)}(r^2, t)$ defined by Eq. (3.5). This follows from the facts that the form of the series expansion for $E(r^2, t, s)$ and $E(r^2, t, s; 2)$ are the same and these functions satisfy the same differential equation and initial conditions. Hence, the series (3.11) converges when $n = 2$. Now define new functions $c^{(k)}(r^2, t; n)$ by the formula

$$c^{(k)}(r^2, t; n) = \frac{2e^{(k)}(r^2, t; n)\, \Gamma(k + n/2 - \frac{1}{2})}{\Gamma(n/2 - \frac{1}{2})\, \Gamma(k)}; \quad k \geqslant 1. \tag{3.14}$$

Then from Eq. (3.12) and (3.13) it is seen that the $c^{(k)}(r^2, t; n)$ satisfy the recursion formula

$$\begin{aligned} c_r^{(1)} &= -trA - (t^2/2)\, rB \\ 2(k-1)\, c_r^{(k)} &= (2k-3)\, c_r^{(k-1)} - rc_{rr}^{(k-1)} - rAc^{(k-1)} - rB \int_0^t c^{(k-1)}\, d\tau; k \geqslant 2. \end{aligned} \tag{3.15}$$

and the initial conditions

$$c^{(k)}(0, t; n) = 0; \qquad k \geqslant 1. \tag{3.16}$$

Equations (3.15) and (3.16) imply that the $c^{(k)}(r^2, t; n)$ are in fact independent of n. Since we know the series (3.11) is convergent when $n = 2$, we can now conclude from Eq. (3.14) and the fact that the $c^{(k)}(r^2, t; n)$ are independent of n that the series (3.11) converges absolutely and uniformly for r and t arbitrarily large (but bounded) and $|s| \leqslant 1$. This establishes the existence of the operator defined by Eqs. (3.9) and (3.11).

Motivated again by the results of section 2, we differentiate the representation (3.9) with respect to t and define a new operator mapping solutions of equation (1.6) onto solutions of Eq. (1.5). If in this operator we now set

$$h(\mathbf{x}, t) = \int_0^1 s^{n-2} H(\mathbf{x}(1 - s^2), t) \frac{ds}{(1 - s^2)^{1/2}}, \tag{3.17}$$

we arrive (c.f. [6, 10]) at the following integral operator which maps real valued solutions of Eq. (1.6) which vanish at $t = 0$ into the class of real valued solutions of equation (1.5) which vanish at $t = 0$ (we again assume $h(\mathbf{x}, t)$ and $u(\mathbf{x}, t)$ are defined in a domain of the form $D \times T$ where D is starlike with respect to the origin):

$$u(\mathbf{x}, t) = h(\mathbf{x}, t) + \int_0^t \int_0^1 \sigma^{n-1} G_t(r^2, 1 - \sigma^2, t - \tau)\, h_\tau(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau. \tag{3.18}$$

In Eq. (3.18) $G(r^2, \rho, t)$ is defined by Eq. (3.8) and is independent of n. This is the basis for refering to the approach used in this section as a "method of ascent." Such techniques were first used by R. P. Gilbert [10, 11] (see also [2, p. 68]) in his investigation of the elliptic equation

$$\Delta_n u + A(r^2)u = 0. \tag{3.19}$$

Subsequently this approach was extended by Colton and Gilbert [6] to treat the fourth order elliptic equation

$$\Delta_n^2 u + A(r^2)\,\Delta_n u + B(r^2)u = 0. \tag{3.20}$$

We now want to show that the operator defined by Eq. (3.18) is invertible, i.e. Eq. (3.18) defines an operator which maps solutions of Eq. (1.6) which vanish at $t = 0$ onto the class of solutions of Eq. (1.5) which vanish at $t = 0$. To this end we differentiate Eq. (3.18) with respect to t and rewrite the resulting expression as the Volterra integral equation

$$\begin{aligned}\Phi_t(r; \theta; \phi, t) &= \psi_t(r; \theta; \phi, t) + \int_0^r K^{(1)}(r, \rho, 0)\, \psi_t(\rho; \theta; \phi, t)\, d\rho \\ &\quad + \int_0^t \int_0^r K^{(2)}(r, \rho, t-\tau)\, \psi_t(\rho; \theta; \phi, \tau)\, d\rho\, d\tau, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \Phi(r; \theta; \phi, t) &= r^{(n-2)/2} u(r; \theta; \phi, t), \\ \psi(r; \theta; \phi, t) &= r^{(n-2)/2} h(r; \theta; \phi, t), \\ K^{(1)}(r, \rho, 0) &= (1/2r)\, G_t(r^2, 1-(\rho/r), 0), \\ K^{(2)}(r, \rho, t) &= (1/2r)\, G_{tt}(r^2, 1-(\rho/r), t), \end{aligned} \tag{3.22}$$

and $(r; \theta; \phi)$ are spherical coordinates. From the recursion formula (3.15) it is seen that each $c^{(k)}(r^2, t; n)$ is of the form

$$c^{(k)}(r^2, t; n) = r^{2k} \tilde{c}^{(k)}(r^2, t; n), \tag{3.23}$$

where $\tilde{c}^{(k)}(r^2, t; n)$ is an entire function of r^2 and t. This follows from the fact that the differential operator $(2k-3)(d/dr) - r(d^2/dr^2)$ annihilates r^{2k-2}. Hence, the functions $K^{(1)}(r, \rho, 0)$ and $K^{(2)}(r, \rho, t)$ defined in Eq. (3.22) are entire functions of r and t and analytic in ρ for $|\rho| \leqslant |r|$. Equations of the form of Eq. (3.21) have previously been studied by Vekua (c.f. [21, pp. 11–16]), and using his techniques we can easily show that for each continuous function $\Phi_t(r; \theta; \phi, t)$ there exists a unique continuous solution $\psi_t(r; \theta; \phi, t)$ of the integral equation (3.21). Furthermore, it can be verified without excessive hardship that if $r^{(2-n)/2}\, \Phi(r; \theta; \phi, t)$ is a (strong) solution of Eq. (1.5) which vanishes at $t = 0$ then the function $r^{(2-n)/2}\psi(r; \theta; \phi, t)$, where $\psi(r; \theta; \phi, t)$ is a solution of the integral equation (3.21), is a (strong) solution of Eq. (1.6) which vanishes at $t = 0$. We can now conclude that the operator defined by Eq. (3.18) is invertible, since if $\Phi(r; \theta; \phi, 0) = 0$ then $\Phi(r; \theta; \phi, t)$ is uniquely determined by $\Phi_t(r; \theta; \phi, t)$. (In the case when $A(r^2) \leqslant 0$ an alternate proof of the invertibility of the operator defined in Eq. (3.18) can be deduced from the results of Section 4 of this paper.)

In passing we note that by comparing Eqs. (3.8), (3.12), and (3.14) with the corresponding equations for Gilbert's G-function for Eq. (3.19) [10] we have that

$$G_t(r^2, \rho, 0) = \tilde{G}(r, \rho), \tag{3.24}$$

where $\tilde{G}(r, \rho)$ denotes Gilbert's G-function for Eq. (3.19). In particular, Gilbert's method of ascent for elliptic equations appears as the limiting case of Eq. (3.18) obtained by differentiating both sides of this equation with respect to t and then letting t tend to zero.

We summarize our results in the following theorem.

THEOREM 3.1. *Let $u(\mathbf{x}, t)$ be a real valued strong solution of Eq.* (1.5) *defined in a domain $D \times T$ where D is starlike with respect to the origin and suppose $u(\mathbf{x}, 0) = 0$. Then $u(\mathbf{x}, t)$ can be represented in the form of Eq.* (3.18) *where $h(\mathbf{x}, t)$ is a solution of Eq.* (1.6) *such that $h(\mathbf{x}, 0) = 0$ and $G(r^2, \rho, t)$ is defined by Eq.* (3.8). *$G(r^2, \rho, t)$ is an entire function of r^2 and t and is analytic for $| \rho | \leqslant 1$. $G_t(r^2, \rho, 0) = \tilde{G}(r, \rho)$ where $\tilde{G}(r, \rho)$ is Gilbert's G-function for the elliptic equation* (3.19).

COROLLARY. *Let $u(\mathbf{x}, t)$ be a real valued strong solution of Eq.* (1.5) *defined in a domain $D \times T$ where D is starlike with respect to the origin and suppose $u(\mathbf{x}, 0) = 0$. Then, for each fixed t, $u(\mathbf{x}, t)$ is an analytic function of the variables $x_1, \ldots, x_n$ for $\mathbf{x} \in D$.*

Proof of Corollary. This follows from the representation (3.18), the analyticity of $G(r^2, \rho, t)$, and the fact that if $h(\mathbf{x}, t)$ is a solution of Eq. (1.6) in $D \times T$ and vanishes at $t = 0$, then $h(\mathbf{x}, t)$ is a harmonic function of $x_1, \ldots, x_n$ in D (for each fixed t), and, hence, is analytic in these variables.

IV. THE FIRST INITIAL-BOUNDARY VALUE PROBLEM

The (first) initial-boundary value problem for Eqs. (1.4) and (1.5) is to find a strong solution of the differential equation in $D \times T$ (where D is bounded, simply connected and has Lyapunov boundary ∂D) which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values on $\partial D \times T$. From the results of [17] such a solution exists and is unique provided $c(x, y) \leqslant 0$ and $A(r^2) \leqslant 0$ respectively in the closure of D. Our purpose is to reformulate this initial-boundary value problem into the problem of solving an integral equation whose form is suitable for solution by iteration or numerical methods, thus giving a constructive method of exhibiting the solution.

We first consider Eq. (1.4). In [5] we have defined a fundamental solution of Eq. (1.4) to be a function of the form

$$S(x, y, t; \xi, \eta, \tau) = A(x, y, t; \xi, \eta, \tau) \log(1/r) + B(x, y, t; \xi, \eta, \tau), \tag{4.1}$$

where $A(x, y, t; \xi, \eta, \tau) = \tilde{A}(z, z^*, t; \zeta, \zeta^*, \tau)$ and $B(x, y, t; \xi, \eta, \tau) = \tilde{B}(z, z^*, t; \zeta, \zeta^*, \tau)$ have the series expansions

$$\begin{aligned} \tilde{A}(z, z^*, t; \zeta, \zeta^*, \tau) &= \sum_{j=1}^{\infty} \tilde{A}^{(j)}(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!}, \\ \tilde{B}(z, z^*, t; \zeta, \zeta^*, \tau) &= \sum_{j=1}^{\infty} \tilde{B}^{(j)}(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!}, \end{aligned} \tag{4.2}$$

which converge absolutely and uniformly for arbitrary values of t and τ and $z, \zeta \in \Omega$, z^*, $\zeta^* \in \Omega^*$ where $\zeta = \xi + i\eta$, $\zeta^* = \xi - i\eta$, $z = x + iy$, $z^* = x - iy$, Ω is an arbitrary compact subset of the complex plane, and $\Omega^* = \{z^*: \bar{z}^* \in \Omega\}$. The coefficients $\tilde{A}^{(j)}(z, z^*; \zeta, \zeta^*)$ and $\tilde{B}^{(j)}(z, z^*; \zeta, \zeta^*)$ can be determined recursively and satisfy (among other initial conditions)

$$\begin{aligned} &\tilde{A}^{(1)}(\zeta, \zeta^*; \zeta, \zeta^*) = 1, \\ &\tilde{A}^{(j)}(\zeta, \zeta^*; \zeta, \zeta^*) = 0 \qquad \text{for} \quad j \geqslant 2, \\ &\tilde{B}^{(j)}(\zeta, \zeta^*; \zeta, \zeta^*) = 0 \qquad \text{for} \quad j \geqslant 1. \end{aligned} \tag{4.3}$$

Motivated by the use of double layer potentials to solve the Dirichlet problem for elliptic equations, we look for a solution of the first initial-boundary value problem in the form

$$u(x, y, t) = \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^2}{\partial \nu \, \partial \tau} S(\xi, \eta, \tau; x, y, t) \, ds \, d\tau, \tag{4.4}$$

where $\mu(\xi, \eta, \tau)$ is a potential to be determined, ν is the inner normal to ∂D, and $ds\, d\tau$ is an element of surface area of $\partial D \times T$. Since as a function of its last three variables $S(\xi, \eta, \tau; x, y, t)$ is a solution of Eq. (1.4) and $S_\tau(\xi, \eta, t; x, y, t)$ is a fundamental solution of $\Delta_2 u + c(x, y)u = 0$, which is independent of t, it is easily verified that if $\mu(\xi, \eta, \tau)$ is continuous in the closure of $D \times T$ then Eq. (4.4) defines a strong solution of Eq. (1.4) which is continuously differentiable with respect to t in the closure of $D \times T$. Now suppose we want to determine $\mu(\xi, \eta, \tau)$ such that for (x, y, t) on $\partial D \times T$ we have

$u(x, y, t) = f(x, y, t)$, where $f(x, y, t)$ is a prescribed function on $\partial D \times T$ which is continuously differentiable with respect to t. Differentiating Eq. (4.4) with respect to t, letting (x, y, t) approach $\partial D \times T$, and using the well known properties of logarithmic potentials (c.f. [8, pp. 334–339]), leads to the following integral equation for $\mu(\xi, \eta, \tau)$:

$$\begin{aligned} f_t(x, y, t) = \mu(x, y, t) &+ \frac{1}{\pi} \int_{\partial D} \mu(\xi, \eta, t) \frac{\partial^2}{\partial \nu \, \partial \tau} S(\xi, \eta, t; x, y, t) \, ds \\ &+ \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^3}{\partial \nu \, \partial \tau \, \partial t} S(\xi, \eta, \tau; x, y, t) \, ds \, d\tau. \end{aligned} \tag{4.5}$$

Note that no residue arises from the second integral in Eq. (4.5) as (x, y, t) approaches $\partial D \times T$ due to the conditions imposed by Eq. (4.3).

We will now show that the integral Eq. (4.5) can always be solved for $\mu(x, y, t)$ provided that $f_t(x, y, t)$ is continuous and $c(x, y) \leqslant 0$ for (x, y) in the closure of D. Equation (4.5) can be written in the form

$$f_t = (\mathbf{I} + \mathbf{T})\mu + \mathbf{L}\mu, \tag{4.6}$$

where

$$\begin{aligned} \mathbf{T}\mu &= \frac{1}{\pi} \int_{\partial D} \mu(\xi, \eta, t) \frac{\partial^2}{\partial \nu \, \partial \tau} S(\xi, \eta, t; x, y, t) \, ds \\ \mathbf{L}\mu &= \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^3}{\partial \nu \, \partial \tau \, \partial t} S(\xi, \eta, \tau; x, y, t) \, ds \, d\tau. \end{aligned} \tag{4.7}$$

Note that $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator with a continuous kernel (due to Eqs. (4.1)–(4.3)). Since $S_\tau(\xi, \eta, t; x, y, t)$ is a (normalized) fundamental solution for the equation $\Delta_2 u + c(x, y)u = 0$ and $c(x, y) \leqslant 0$, the operator $(\mathbf{I} + \mathbf{T})^{-1}$ exists (c.f. [8, pp. 364–365]). Furthermore, by Fubini's theorem, $\mathbf{L}$ and $\mathbf{T}$ commute (and, hence, so do $\mathbf{L}$ and $(\mathbf{I} + \mathbf{T})^{-1}$), and due to $\mathbf{L}$ being a Volterra operator, $\|(\mathbf{I} + \mathbf{T})^{-m}\mathbf{L}^m\| < 1$ for m sufficiently large ($\| \cdot \|$ denotes the L_2 operator norm). Thus, the operator $(\mathbf{I} + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L})^{-1}$ exists. Hence, from Eq. (4.6) we have

$$(\mathbf{I} + \mathbf{T})^{-1} f_t = \mu + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L}\mu \tag{4.8}$$

and

$$\begin{aligned} \mu &= (\mathbf{I} + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L})^{-1}(\mathbf{I} + \mathbf{T})^{-1} f_t \\ &= (\mathbf{I} + \mathbf{T} + \mathbf{L})^{-1} f_t \,. \end{aligned} \tag{4.9}$$

The continuity of f_t implies that μ is continuous in the closure of $D \times T$, and, hence, Eqs. (4.4) and (4.9) give the desired solution of the first initial-boundary value problem.

Equation (4.5) lends itself to various procedures for approximating solutions of the first initial-boundary value problem for Eq. (1.4). For example possible approaches would be to replace the integral equation (4.5) by a system of algebraic equations (c.f. [12, pp. 98–103]) or to use the method of moments (c.f. [12, pp. 150–154]). However, the systematic study of integral equations of mixed Fredholm and Volterra type such as the ones arising here has (to the author's knowledge) not yet been undertaken by mathematicians working in the area of integral equations or numerical analysis. It would be desirable to complete such an investigation and hopefully this paper will give some motivation for mathematicians to begin working on integral equations of this type.

We summarize the results obtained up to this point in the following theorem.

THEOREM 4.1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ with Lyapunov boundary ∂D and $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Assume that $c(x, y) \leqslant 0$ in the closure of D. Then Eqs.* (4.4) *and* (4.9) *define the* (*unique*) *strong solution to Eq.* (1.4) *in $D \times T$ which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values $f(x, y, t)$ on $\partial D \times T$.*

We now turn our attention to developing a constructive method for solving the first initial-boundary value problem for Eq. (1.5) in the case when $n > 2$. A straightforward generalization of the analysis just completed for the case of Eq. (1.4) is no longer possible since a fundamental solution for pseudoparabolic equations in more than two space dimensions has not yet been constructed. Motivated by the work of Gilbert [10, 11] we will overcome this problem through the use of Theorem 3.1 and the well known fundamental solution for Laplace's equation. In the following discussion we will assume that the domain D, in addition to the hypothesis given at the beginning of this section, is also starlike with respect to the origin.

We begin by differentiating both sides of Eq. (3.18) with respect to t to arrive at the general representation

$$u_t(\mathbf{x}, t) = h_t(\mathbf{x}, t) + \int_0^1 \sigma^{n-1} G_t(r^2, 1 - \sigma^2, 0)\, h_t(\mathbf{x}\sigma^2, t)\, d\sigma + \int_0^t \int_0^1 \sigma^{n-1} G_{tt}(r^2, 1 - \sigma^2, t - \tau)\, h_\tau(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau. \quad (4.10)$$

Since $h(\mathbf{x}, t)$ is a solution of Eq. (1.6), it is clear that, for each fixed t, $h_t(\mathbf{x}, t)$ is harmonic. Hence, for $n > 2$ we can represent $h_t(\mathbf{x}, t)$ as a double layer potential

$$h_t(\mathbf{x}, t) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial D} \mu(\mathbf{y}, t) \frac{\partial}{\partial \nu} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^{n-2}} \right) ds, \quad (4.11)$$

where ν is the inner normal on ∂D, $\mu(\mathbf{y}, t)$ is a potential to be determined, and $\mathbf{x} \in D$ (for $n = 2$ we would represent $h_t(\mathbf{x}, t)$ as a double layer logarithmic potential). Substituting (4.11) into Eq. (4.10), interchanging orders of integration, and letting $\mathbf{x}$ approach the boundary of D, leads to the following integral equation for the determination of $\mu(\mathbf{x}, t)$ (where $u(\mathbf{x}, t) = f(\mathbf{x}, t)$ on ∂D):

$$f_t(\mathbf{x}, t) = \mu(\mathbf{x}, t) + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial D} \mu(\mathbf{y}, t)\, K^{(1)}(\mathbf{x}, \mathbf{y}, t)\, ds + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^t \int_{\partial D} \mu(\mathbf{y}, \tau)\, K^{(2)}(\mathbf{x}, \mathbf{y}, t - \tau)\, ds\, d\tau, \tag{4.12}$$

where

$$\begin{aligned} K^{(1)}(\mathbf{x}, \mathbf{y}, t) &= \frac{\partial}{\partial \nu} \left(\frac{1}{|\,\mathbf{x} - \mathbf{y}\,|^{n-2}} \right) + \int_0^1 \sigma^{n-1} G_t(r^2, 1 - \sigma^2, 0) \frac{\partial}{\partial \nu} \left(\frac{1}{|\,\mathbf{x}\sigma^2 - \mathbf{y}\,|^{n-2}} \right) d\sigma \\ K^{(2)}(\mathbf{x}, \mathbf{y}, t) &= \int_0^1 \sigma^{n-1} G_{tt}(r^2, 1 - \sigma^2, t) \frac{\partial}{\partial \nu} \left(\frac{1}{|\,\mathbf{x}\sigma^2 - \mathbf{y}\,|^{n-2}} \right) d\sigma. \end{aligned} \tag{4.13}$$

We note that the kernels $K^{(1)}(\mathbf{x}, \mathbf{y}, t)$ and $K^{(2)}(\mathbf{x}, \mathbf{y}, t)$ have weak singularities at $x = y$. Hence, Eq. (4.12) is again of the form

$$f_t = (\mathbf{I} + \mathbf{T})\mu + \mathbf{L}\mu, \tag{4.14}$$

where $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator. From Theorem 3.1 it is seen that the operator $\mathbf{I} + \mathbf{T}$ is identical with the operator defined in Eq. (4.42) of [10], and, hence, if $A(r^2) \leqslant 0$ in the closure of D, $(\mathbf{I} + \mathbf{T})^{-1}$ exists. Repeating the analysis which led to Eq. (4.9) we have that $(\mathbf{I} + \mathbf{T} + \mathbf{L}^{-})^{1}$ exists and

$$\mu = (\mathbf{I} + \mathbf{T} + \mathbf{L})^{-1} f_t\,. \tag{4.15}$$

Equations (4.15), (4.10), and (4.11) now give the solution of the first initial-boundary value problem for Eq. (1.5). We summarize this result in the following theorem.

THEOREM 4.2. *Let D be a bounded domain in $\mathbb{R}^n$, $n > 2$, which is starlike with respect to the origin and has Lyapunov boundary ∂D and let $T = \{t\colon 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Assume that $A(r^2) \leqslant 0$ in the closure of D. Then Eqs.* (4.10), (4.11), *and* (4.15) *define the* (*unique*) *strong solution to Eq.* (1.5) *in $D \times T$ which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values $f(\mathbf{x}, t)$ on $\partial D \times T$.*

In closing we would like to point out that the assumption made throughout this paper that the coefficients of Eqs. (1.4) and (1.5) are entire functions can obviously be weakened to the requirement that these coefficients only be analytic in a sufficiently large ball about the origin.

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